Relativity Lecture 2: Intervals

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KMHS Physics Club, February 6 2018

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Moving with light

You probably already know that only massless objects move at c (in fact they are confined to this speed).
Remember that it only makes sense to talk about *relative* velocity. If I wanted to move at c, couldn't I just look at a photon? The photon is moving at speed c, and I am at rest relative to it, so from it's point of view, I would be moving at c!

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- By writing precisely what is going on, we can find out the issue
- In our proper frame, we are not moving, and we observe the photon to be moving at speed c
- In the photon's proper frame, it is not moving, and it sees us as moving at speed c
- No proper frame exists for a photon (second postulate)

Matrix Multiplication

A linear transformation T on the vector space $\mathbb{R}^n$ satisfies A $n \times m$ matrix transforms vectors in $\mathbb{R}^m$ into vectors in $\mathbb{R}^n$. So for example,

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 12 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \\ 12 \end{pmatrix} + 6 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 28 \\ 17 \\ 60 \end{pmatrix}.$$

The *nth* column of a matrix shows where e_n transforms to. So $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ can really be decomposed into $5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 5e_1 + 6e_2$, and then we can apply A to each basis vector individually.

Dot Product

The dot product between two vectors $x = x^\mu$ and $y = y^\mu$ is

$$\sum_{k=1}^{n} x^k y^k = x^\mu y^\mu$$

where summation over μ is implied over repeated indices (Einstein summation convention). For example the dot product in $\mathbb{R}^3$ looks like

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 2 + 6 + 4 = 12.$$

The dot product can be written in $\mathbb{R}^n$ as

$$x \cdot y = x^T y$$

so one can dispense with the dot product and redefine it as a linear transformation x^T acting on y. So from the previous example

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (2, 3, 4) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Tensors

The dual space of the vector space V is denoted by V^V and is the set of all linear maps from V to $\mathbb{R}$. In other words, it defines a dot product where the product

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A tensor is a generalization of a matrix. Previously we defined a linear transformation, represented by a matrix, to be a mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Tensors are mappings between vector spaces or between dual vector spaces, or between both.

Inner Product

The generalization of the dot product is called the inner product and satisfies the following

1. $\langle v + u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$
2. for scalar α, $\langle \alpha v, w \rangle = \langle v, \alpha w \rangle = \alpha \langle v, w \rangle$
3. $\langle v, w \rangle = \langle w, v \rangle$

Invariant Interval

We have not established a general framework (the problems are difficult mainly because you use heuristics) The goal of this lecture is to establish precisely how a set of coordinates transforms from one reference frame to another Along the way, we will learn about velocity addition, spacetime diagrams, momentum, and optical effects

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Consider the displacement of light, from point P_1 to P_2. Using the above equation,

$$c\Delta t = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \implies c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

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which is something everyone can agree on. Note: One may use either the differential or displacement form. So if we assume that the initial coordinates is null,

$$c^2 t^2 - x^2 - y^2 - z^2 = 0.$$

Furthermore, the c is becoming very annoying, so one can switch to more natural units with $c = 1$ so that

$$t^2 - x^2 - y^2 - z^2 = 0.$$

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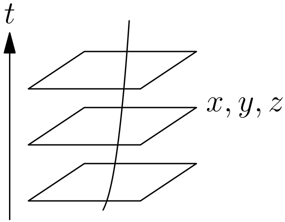
and call it the *interval*. The interval between two events is the distance between them within the four-dimensional space with axes t, x, y, z (note that the geometry of this space is not euclidean, so distances are not defined as normal). Events are called *world points* and the paths they trace out are called *world lines*.



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$$ds^2 = a ds'^2$$

for some constant a. Now all we need to show is $a = 1$.

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$$ds^2 = a(v_1)ds_1^2 = a(v_2)ds_2^2, \quad ds_1^2 = a(v_{12})ds_2^2 \implies \frac{a(v_2)}{a(v_1)} = a(v_{12}).$$

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This does not make sense because we could have the frames moving at different angles to each other, but this formula does not depend on angles.The symmetrical way of resolving this is $a(v) = 1$ for all v. Thus $ds^2 = ds'^2$. *The interval is a measurement that everyone can agree on.*

Lightlike

There are three cases: $s^2 < 0, s^2 = 0, s^2 > 0$.

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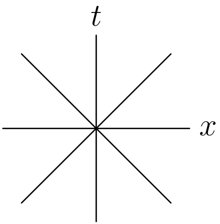
$$t^2 = x^2$$

gives



The cone above is called the light cone. Event 1 is the origin and event 2 lies somewhere on the lines $t = \pm x$.

Timelike

$$s^2 > 0 \implies t^2 - x^2 - y^2 - z^2 > 0 \implies t^2 > l^2$$

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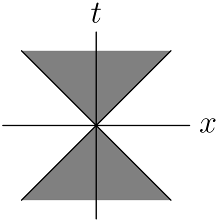

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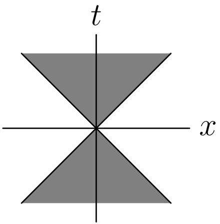

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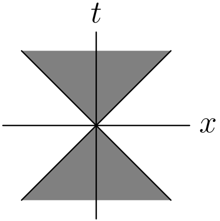

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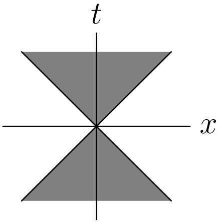
Now it is pretty clear what this means; the distance that light travels during the time between the events is greater than the spacial distance between the events. Two such events are called *timelike*.

Timelike



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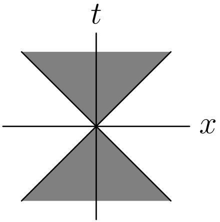

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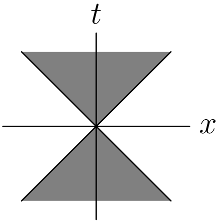

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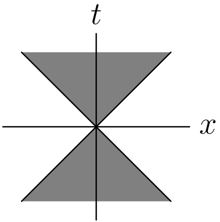
Timelike events always occur at different times. Furthermore, an event in the top cone occurs after event 1 in all reference frames (absolute future), and an event in the bottom cone occurs before event 1 in all reference frames (absolute past).

Spacelike

$$(ct)^2 < l^2$$

so even light does not have enough time to travel the distance between the two events. This is the white area in



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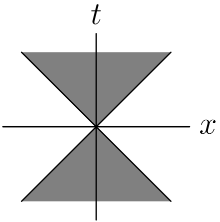

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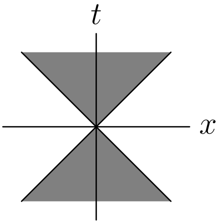

Spacelike events are always in different locations (why?). There exists a frame in which the events are simultaneous. All events in the exterior of the light cone cannot affect event 1.

Classification via inner products

We shall rename the coordinates:

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Any 1-tensor A^μ that transforms in a way that keeps the its components invariant in the above way is called a *contravariant 4-vector*. We define a Lorentzian inner product so that

$$\langle A, A \rangle = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2.$$

We can create a more convenient notation by defining

$$A_0 = A^0, \quad A_i = -A^i,$$

so that we can write

$$A^\mu A_\mu.$$

Boosts

There is an additional (temporal) dimension around which one can perform a rotation. This is not a pure spatial rotation like considered before, so it is called a *boost*.

$$\begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}.$$

From this we get the two equations

$$ct\cosh\psi + x\sinh\psi = ct', \quad ct\sinh\psi + x\cosh\psi = x'.$$

We can check that this boost keeps the interval invariant.

Invariance of interval under boost

$$
\begin{aligned}
c^2t^2 - x^2 - y^2 - z^2 =& c^2t'^2 - x'^2 - y'^2 - z'^2 \\
=& c^2t'^2 - x'^2 - y'^2 - z'^2 \\
=& (ct\cosh\psi + x\sinh\psi)^2 - (ct\sinh\psi + x\cosh\psi)^2 - y'^2 - z'^2 \\
=& (ct)^2\cosh^2\psi + x^2\sinh^2\psi + 2ctx\cosh\psi\sinh\psi \\
& - (ct)^2\sinh^2\psi - x^2\cosh^2\psi - 2ctx\sinh\psi\cosh\psi - y^2 - z^2 \\
=& (ct)^2(\cosh^2\psi - \sinh^2\psi) + x^2(\cosh^2\psi - \sinh^2\psi) - y^2 - z^2 \\
=& c^2t^2 - x^2 - y^2 - z^2
\end{aligned}
$$

Lorentz Transformations

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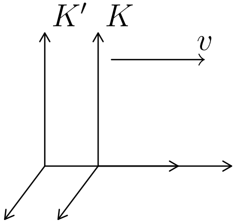
- Consider the origin of the K system.



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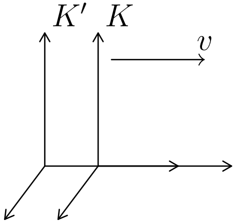

- $x = 0$ and $x' = vt'$. Substituting these values into the equations, we get:

$$ct \cosh \psi = ct', \quad ct \sinh \psi = vt',$$

and dividing the second equation by the first we get

$$\frac{v}{c} = \tanh \psi \implies \sinh \psi = \frac{v}{c} \cosh \psi.$$

Lorentz Transformations

- Now using the classic hyperbolic identity $\cosh^2\psi - \sinh^2\psi = 1$, we get

$$\cosh^2\psi - \left(\frac{v}{c}\cosh\psi\right)^2 = 1 \implies \cosh\psi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

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- Now all that remains to do is sub these back into the original rotation equations, to get

$$ct' = \gamma\left(ct + \frac{xv}{c}\right), \gamma x' = \gamma(tv + x).$$

Lorentz Transformations

Thus the Lorentz Transformation equations are

$$
\begin{aligned}
t' &= \gamma\left(t + \frac{xv}{c^2}\right) \\
x' &= \gamma(x + tv) \\
y' &= y \\
z' &= z
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
t' &= \gamma(t + xv) \\
x' &= \gamma(x + tv) \\
y' &= y \\
z' &= z
\end{aligned}
\tag{2}
$$

in natural units

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- Note that this looks like the formula for addition in tanh :

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + (\tanh x)(\tanh y)}$$