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# Chapter 1

# Relativistic Electrodynamics

## § 1 Basics of Special Relativity

**Solution: (*Blake Law and Noah Law*) The Barn Paradox**

*Farmer's Perspective*: From the farmer's perspective, who is stationary, which we will describe as frame F, he will see an apparent length contraction of the pole as follows, defining $x_2^{'}$ and $x_1^{'}$ as the end and starting point of the pole of length $L$ (where L is the length of barn ($l$) plus some positive length) in the moving frame F[1]:

$$L = x_2^{'} - x_1^{'} = \frac{x_2 - vt_2 - x_1 + vt_1}{\gamma}$$

$$L = \frac{L_0}{\gamma}$$

For relatively large values of v, this contraction will become apparent to the farmer, being able to fully fit the pole inside the barn all at the same time in his perspective and reference frame and subsequently leaving all in one piece.

*Runner's Perspective*: A length contraction of the barn (length $l$) will occur for the runner, modeled by the following equation:

$$l' = l\sqrt{1-\beta^2} = \frac{l_0}{\gamma}$$

From the runner's perspective, which we will describe as in frame F', the Lorentz transformation of time is as follows, where $t$ is the measured time by the moving frame and $x$ is the distance of some object from the runner:

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}} = \gamma(t - \frac{vx}{c^2})$$

[1]HyperPhysics: *Georgia State University*. Lorentz Transformation. Accessed 25 November 2017.

From this we can conclude in the runner's perspective, he will enter the barn with the furthest right door closing and the left door open, because the distance $x$ is greater for the furthest right door, making the quantity $t'$ smaller. As the runner moves through the barn, the right door opens allowing for the right end of the pole to exit the barn. As the left end of the pole clears the left door to the barn, the left door closes. The event of the doors closing does not occur at the same time as evidenced by relevant equations and the fact that if they did, the pole would be unable to go through (as it would be crushed). This demonstrates the fact that changing reference frames does not guarantee identical results.

Therefore, this 'paradox' can be explained by the fact that there is a loss of simultaneity because of the magnitude of speed of the runner, or that from one perspective time is dilated and the barn is contracted, where the doors close and different times, while from the other perspective the length of the pole is contracted, seemingly not hitting the walls of the barn.

**Solution: (*Dimitrios*) Expansion of the Spacetime Interval and Proof of its Invariance**

Let $s^2$ be the spacetime interval between two events in the frame $K$ and let $s'^2$ be the spacetime interval between the same two events in the proper frame $K'$, which is inertial to the frame $K$.

$$\begin{aligned} ds^2 &= dx_\mu dx^\mu \\ &= \begin{bmatrix} -cdt & dx & dy & dz \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

We use the Lorentz transformation equations on $ds'$ to show that $ds'^2 = ds^2$.

$$\begin{aligned} ds^2 &= -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \\ &= -\gamma^2(cdt - \beta dx)^2 + \gamma^2(dx - vdt)^2 + dy^2 + dz^2 \\ &= \gamma^2(v^2dt^2 + dx^2 - \frac{v^2}{c^2}dx^2 - c^2dt^2) + dy^2 + dz^2 \\ &= \frac{c^2}{c^2 - v^2}\left(dt^2(v^2 - c^2) + \frac{dx^2}{c^2}(c^2 - v^2)\right) + dy^2 + dz^2 \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2. \end{aligned}$$

**Solution: (*Dimitrios*) Time Dilation**

Let $dx' = dy' = dz' = 0$ since there is no change in the physical location of the events in the proper frame $K'$. By the equality of the spacetime interval, we have

$$-c^2dt'^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

The inverse Lorentz transformation equations can be found by taking the inverse of the matrix

$$\begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}^{-1} = \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix}.$$

Note that $dy = dy'$ and $dz = dz'$. We apply these equations to the right hand side of (1):

$$\begin{aligned} -c^2dt'^2 &= -c^2dt^2 + \gamma^2(dx' + \beta c dt')^2 \\ -c^2dt'^2 &= -c^2dt^2 + \gamma^2\beta^2c^2dt'^2 \\ dt'^2 &= dt^2 - \gamma^2\beta^2dt'^2 \\ dt'^2(1+\gamma^2\beta^2) &= dt^2 \\ dt'\sqrt{1+\gamma^2\beta^2} &= dt \\ dt'\sqrt{\frac{1-\beta^2+\beta^2}{1-\beta^2}} = \gamma dt' &= dt \end{aligned}$$

Since $\gamma \geq 1$ for $v \leq c$, the equation $\gamma dt' = dt$ implies that the a person in the $K$ frame experience a longer period of time than the proper time of the $K'$ frame; if the $K$ person were to compare his clocks with the clock in the $K'$ frame, he would find that it is slow. Clocks in Tokyo and Kyoto are synchronized. If you travel on a bullet train from Kyoto to Tokyo, as you step out of the train you will need to increase the time on your watch to match the time in Tokyo.

**Solution: (*Dorian*) Length Contraction**

We will use a thought experiment to derive length contraction. Let there be a perfectly cylindrical pole travelling at some speed $v$ relative to an observer. At some time $t$, this observer records the location of the ends of the pole at $x_1'$ and $x_2'$ and uses that data to find the pole's length, $x_2' - x_1'$. Meanwhile, another observer travelling at the same speed as the perfectly cylindrical pole records the location of the ends of the pole at $x_1$ and $x_2$ and uses that information to find the pole's proper length, $x_2 - x_1$. In order to derive the formulas for length contraction , we use

$$\begin{aligned} x_1 &= \gamma(x_1' + vt_1') \\ x_2 &= \gamma(x_2' + vt_2') \end{aligned}$$

, subtracting the first equation from the second, and obtain $x_2 - x_1 = \gamma(x_2' - x_1' + vt_1' - vt_2')$. Since we are measuring distance at the same time, $t_1' = t_2'$, so $x_2 - x_1 = \gamma(x_2' - x_1')$, which can be rewritten as $L_p = \gamma L$. To see this effect, let observer one be travelling on a space ship at $.5c$ between Planet Alpha and Planet Omicron, which are stationary relative to each other, and let observer two be stationary on Planet Alpha. Both attempt to measure the distance between Planet Alpha and Planet Omicron. Observer two times how long it takes observer one to get to Planet Omicron from Planet Alpha and measures it to be 2 hours. Therefore, he measures the distance to be $.5c \cdot 2\,\text{hours} \approx 1341233258.77$ miles. Observer one also uses a similar methodology. He finds it takes him $\frac{2\,\text{hours}}{\gamma} \approx 1.73205086$ hours to travel between the two planets. Therefore, he computes the distance to be $1.73205086 \cdot .5c \approx 1161542109.66$ miles. Using the formula for length contraction, we indeed find that $1341233258.77 \approx \gamma 1161542109.66$

**Solution: (*Dorian*) Four-Velocity**

a) Proper time has to be used because time is not an invariant under the Lorentz transformation and because it is consistent through multiple systems. Using nonproper time would result in inconsistencies in translating between systems.

b) Let $(u')^\mu = \left(c, \frac{dx'}{d\tau}, \frac{dy'}{d\tau}, \frac{dz'}{d\tau}\right)$ and $u^\mu = \left(c, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$. Using the Lorentz transformation, we find that

$$(u')^\mu = \left(c, \frac{\frac{dx}{dt} - v}{1 - \frac{vdx}{c^2dt}}, \frac{\gamma dy}{dt}, \frac{\gamma dz}{dt}\right).$$

Let $u = \frac{dx}{dt}$ and $u' = \frac{dx'}{dt'} = \frac{dx'}{d\tau}$. Comparing the two expressions and replacing $\frac{dx}{dt}$ with $u$, we find

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

Through simple algebraic rearrangement, we find

$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}.$$

**Solution: (*Dorian*) Invariance of Energy and Momentum**

We are given that the momentum four-vector is

$$\begin{aligned} p^\mu &= \left(\frac{E}{c}, p_x, p_y, p_z\right) \\ &= m_0 u^\mu \\ &= \left(cm_0, m_0\frac{dx}{d\tau}, m_0\frac{dy}{d\tau}, m_0\frac{dz}{d\tau}\right). \end{aligned}$$

Since we know that the spacetime interval is invariant, and $p = m_0\frac{dx^\mu}{d\tau}$, we are motivated to find the dot product $p \cdot p$. As before, we use the Minkowski metric to change from a contravariant to a covariant vector via the metric of the space. Evaluating and using the previously found expansion of the spacetime interval, we have

$$p_\mu p^\mu = m_0^2 \frac{-c^2dt^2 + dx^2 + dy^2 + dz^2}{d\tau^2}$$

Cancelling, we obtain

$$p_\mu p^\mu = -(m_0c)^2.$$

This expression is of course equivalent to $-\left(\frac{E}{C}\right)^2 + p_x^2 + p_y^2 + p_z^2 = -\left(\frac{E}{C}\right)^2 + p^2$, evaluated using the same Minkowski metric. Rearranging terms, we therefore have

$$E^2 = p^2c^2 + (mc^2)^2$$

**Solution: (*Dimitrios*) Four-Acceleration**

By definition,

$$ds^2 = dx_\mu dx^\mu \implies \frac{dx_\mu}{ds}\frac{dx^\mu}{ds} = 1$$

Since the interval is timelike, $ds = cd\tau$, so $d\tau \propto ds$, which implies that

$$\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = c^2.$$

Taking a derivative over the dot product and noting that the four-acceleration is $w^\mu = \frac{du^\mu}{d\tau}$, we have

$$\frac{d}{d\tau}\left(\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}\right) = 2\frac{dx_\mu}{d\tau}\frac{d^2x^\mu}{d\tau^2} = 0 \implies u_\mu w^\mu = 0.$$

Since we did not involve the mass of the object in the solution, this holds true for an arbitrary mass $m$.

## § 2 Relativistic Electrodynamics and Tensors

**Solution: (*Dimitrios*) The Continuity Equation**

a) The amount of charge in a volume is

$$\int_V \rho d\mathbf{V}.$$

Any charge leaving the volume passes through the surface, and we can write that the current over the closed surface is

$$\oint \vec{j} \cdot d\mathbf{a}.$$

In a closed surface, the outward direction is positive, so the surface integral is positive, but

$$\frac{d}{dt}\int_V \rho d\mathbf{V} = \int_V \frac{\partial\rho}{\partial t} d\mathbf{V}$$

is negative since the amount of charge in the volume is decreasing. By the local conservation of charge, the current flowing through the surface is equal to the negative change in charge in the volume per unit time:

$$\int_V \frac{\partial\rho}{\partial t} d\mathbf{V} = -\oint \vec{j} \cdot d\mathbf{a}.$$

Applying Gauss's theorem on the right hand side, we have

$$\int_V \frac{\partial\rho}{\partial t} d\mathbf{V} = -\oint \vec{\nabla} \cdot \vec{j} d\mathbf{V} \implies \int \left(\vec{\nabla} \cdot \vec{j} + \frac{\partial\rho}{\partial t}\right) d\mathbf{V} = 0.$$

Since this is true for any volume, the integrand is 0, so

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial\rho}{\partial t}.$$

b) The 4-current is $j^\mu = (c\rho, \vec{j})$. The 4-divergence of the 4-current is

$$\partial_\mu j^\mu = \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0,$$

by the continuity equation[2]. This is a statement of charge conservation. The 4-divergence is Lorentz invariant, so we have shown that charge is conserved in any frame of reference.

**Solution: (*Dimitrios*) Maxwell's Equations in Terms of the Potentials**

a) A well known vector calculus identity states that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{Q}) = 0$ (this can be seen since $\vec{\nabla}\times\vec{Q}$ is perpendicular to $\vec{\nabla}$). Maxwell's 2nd equation reads

$$\vec{\nabla}\cdot\vec{B} = 0,$$

which implies that $\vec{B}$ is the curl of some field; there exists a vector field $\vec{A}$ such that

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{2}$$

Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

can be rewritten with $\vec{B} = \vec{\nabla}\times\vec{A}$:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} \implies \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0.$$

Using the well known fact that $\vec{\nabla}\times(\vec{\nabla}A) = 0$ we can write that

$$\vec{E} + \frac{\partial\vec{A}}{\partial t}$$

is the gradient of some vector field, called $-\phi$. Now we have

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \implies \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}. \tag{3}$$

b) Taking the divergence of (3) gives

$$\vec{\nabla}^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\vec{\nabla}\cdot\vec{E} = -\frac{\rho}{\epsilon_0}$$

by Maxwell's first equation (Gauss's law). In order for

$$\vec{\nabla}^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

to look like

$$\Box^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}^2\phi$$

[2] Jackson, J. D. (2013). *Classical Electrodynamics*. Hoboken, NY: Wiley.

we put $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$ (which is allowed due to gauge freedom). Now we have

$$\Box^2\left(\frac{\phi}{c}\right) = \frac{1}{c^3}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c}\vec{\nabla}^2\phi = \frac{\rho}{c\epsilon_0} = \frac{c\rho}{\mu_0}$$

for $\epsilon_0 = \frac{\mu_0}{c^2}$.

Taking the curl of (2) we get

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\phi - \vec{\nabla}^2\vec{A}$$

but since Maxwell's last equation is

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\left(\frac{\vec{j}}{\epsilon_0} + \frac{\partial E}{\partial t}\right)$$

we can isolate $\vec{\nabla}^2\vec{A}$:

$$\vec{\nabla}^2\vec{A} = -\frac{1}{c^2}\left(\frac{\partial}{\partial t}\vec{\nabla}\phi + \frac{\vec{j}}{\epsilon_0} + \frac{\partial E}{\partial t}\right).$$

Now we differentiate (3) to get

$$\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla}\phi - \frac{\partial^2\vec{A}}{\partial t^2} \implies \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{1}{c^2}\left(\frac{\partial\vec{E}}{\partial t} + \frac{\partial}{\partial t}\vec{\nabla}\phi\right).$$

Now we put together the d'Alembertian and notice significant cancellation

$$\Box^2\vec{A} = \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} = \frac{1}{c^2}\left(-\frac{\partial\vec{E}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\phi + \frac{\partial}{\partial t}\vec{\nabla}\phi + \frac{\vec{j}}{\epsilon_0} + \frac{\partial E}{\partial t}\right) = \frac{\vec{j}}{c^2\epsilon_0} = \frac{\vec{j}}{\mu_0}.$$

c) i) We may write the 4-gradient as

$$\partial_\mu = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right).$$

Furthermore, we have

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

so that

$$\partial_\mu\partial^\mu = \partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box^2.$$

Since the d'Alembertian is the contraction of a four-vector, so it is Lorentz invariant. From the continuity equation, we have

$$\partial_\mu j^\mu = 0,$$

which implies that $j^\mu$ is a four-vector. Since, as shown in part d),

$$\Box^2 A^\mu = \frac{j^\mu}{\mu_0}$$

and $\Box^2$ is Lorentz invariant, $A^\mu$ must be a four-vector.

ii) Let the $K$ and $K'$ be two inertial frames. The event $P$ has coordinates $(0,0,0,0)$ and the retarded event $Q$ has coordinates $(ct,x,y,z)$. $r=\sqrt{x^2+y^2+z^2}$ is the distance from $dV$ to the event $P$. When considering light moving between the two events, the interval is zero, and both $t$ and $t'$ are negative since $Q$ is retarded. From the Lorentz transformation equations, we get

$$r' = -ct' = r\gamma\left(-\frac{ct}{r}+\beta\frac{x}{r}\right).$$

Since $\frac{x}{r}$ is $\cos\theta$ where $\theta$ is the angle between $PQ$ and the $x-$axis, we have

$$r' = r\gamma\left(1+\beta\cos\theta\right).$$

Using the Lorentz equations and $dt=-\frac{dr}{c}=-\frac{dx}{c}\cos\theta$ and noting that according to the Lorentz transformations laws, $dV \propto dx$, we have

$$dV' = \gamma dV\left(1+\beta\cos\theta\right).$$

Now dividing the equations for $dV'$ and $r'$ we find that $dV'/r' = dV/r$, proving that $dV/r$ is Lorentz invariant. Since

$$\phi = \int \rho\frac{dV}{r}, \vec{A} = \int \vec{j}\frac{dV}{r},$$

are the components of $A^\mu$, $A^\mu$ must be invariant under the Lorentz transformation, and is thus a 4-vector. [3]

d)

$$\Box^2 A^\mu = \left(\Box^2\frac{\phi}{c}, \Box^2\vec{A}\right) = \left(\frac{c\rho}{\mu_0}, \frac{\vec{j}}{\mu_0}\right) = \frac{j^\mu}{\mu_0}.$$

**Solution: (*Dorian*) Forces in Different Frames**

We begin this problem by noticing that the force component perpendicular to particle's rest frame is $\gamma$ times as great the force component in the unprimed frame. Furthermore, we notice that the force component parallel to the particle's motion is the same. The force in the reference frame of the particle is then $F'_{\|} = qE_{\|}$ while from the unprimed reference frame the force is $F_{e\|} = qE_{\|}$. Looking at the perpendicular force components, we find that $F'_{\perp} = qE'_{\perp} = q\gamma E_{\perp}$ while in the unprimed reference frame, $F_e = \frac{1}{\gamma}q\gamma E_{\perp} = qE_{\perp}$. Therefore $F' = F$ [4].

**Solution: (*Dorian*) Particles in a Wire**

a) The Lorentz force is given by

$$\vec{F} = q_+\vec{E} + q_+\vec{v}\times\vec{B},$$

[3] Fitzpatrick, R. (2006, February 02). *Retarded Potentials.* Retrieved November 26, 2017.

[4] *Electricity and Magnetism* Retrieved December 3, 2017

where $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field. The negative and positive particles in the wire have the same charge density, so the net charge is 0 and $q_+$ feels no electric force. Thus,

$$\vec{F} = q_+\vec{v} \times \vec{B}.$$

Ampere's law for a wire says that[5]

$$\vec{B} = \vec{y}\frac{\vec{I}}{2\pi\epsilon_0 c^2 r}$$

where $\vec{y}$ is a unit vector perpendicular to the vector $\vec{r}$ ($\vec{r}$ is perpendicular to the wire and extends to $q_+$). Now we have, with $\vec{x} = \frac{\vec{v}}{v}$,

$$\vec{F} = q_+\vec{v} \times \vec{B} = \vec{x} \times \vec{y}\frac{\vec{I}qv}{2\pi\epsilon_0 rc^2} = -\vec{z}\frac{\vec{I}qv}{2\pi\epsilon_0 rc^2}.$$

b) First, we need to use the Einstein velocity addition laws developed in 2.1.5 (b) to find the speed of the electrons from the particle's frame of reference. Letting $\beta = \frac{v}{c}$, $\beta_0 = \frac{u}{c}$, and $\beta_0' = \frac{u'}{c}$, we find that

$$\beta_0' = \frac{\beta_0 - \beta}{1 - \beta\beta_0}$$

and

$$\gamma_0' = \gamma\gamma_0(1 - \beta\beta_0).$$

We can now find the linear density of negative charge in the wire in the frame of reference of the particle:

$$\lambda' = \gamma\lambda_0 \frac{\lambda_0}{\gamma_0}\gamma\gamma_0(1 - \beta\beta_0) = \gamma\beta\beta_0\lambda_0.$$

Using Gauss's law, we find

$$E_r' = \frac{\gamma\beta\beta_0\lambda_0}{2\pi\epsilon_0 r'}.$$

Finally, we find the force that the particle experiences to be

$$F_y' = qE_y' = -\frac{q\gamma\beta\beta_0\lambda_0}{2\pi\epsilon_0 r'}$$

considering that the field points in the negative direction at the particle's location [6]. In order to compare with the result from part a, we use $I = -\lambda_0\beta_0 c$, $\beta = \frac{v}{c}$, and $r' = r$ (the distance from the wire to the charge is the same from both the lab's frame of reference and the particle's frame of reference), which gives

$$F_y' = \frac{I\gamma v}{2\pi\epsilon_0 rc^2}.$$

Comparing with the result from part (a), we see the that the force on the particle form the particle's perspective is $\gamma$ times the force on the particle from the lab's perspective. From

[5] Feynman, R. P. (2011). The Feynman lectures on physics (Vol. 2). New York: BasicBooks.
[6] *Electricity and Magnetism* Retrieved December 3, 2017

this, we understand that magnetic and electric forces are both parts of the electromagnetic force, which is invariant under coordinate transformations. Thus, the unification of electricity and magnetism are sound under Einstein's theory of relativity [7].

**Solution: *Dimitrios and Dorian* The Electromagnetic Field Tensor**

*imitrios and Dorian*) Using 3.3.1a, $\vec{B} = \vec{\nabla} \times \vec{A}$. Expanding, we get

$$\vec{B} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right).$$

Since $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$, we expand $\vec{E}$ and get

$$\vec{E} = \left( -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x}, -\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y}, -\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z} \right).$$

Using covariant notation we have

$$\vec{B} = (-(\partial^2 A^3 - \partial^3 A^2), -(\partial^3 A^1 - \partial^1 A^3), -(\partial^1 A^2 - \partial^2 A^1))$$

and

$$\vec{E} = c(-(\partial^0 A^1 - \partial^1 A^0), -(\partial^0 A^2 - \partial^2 A^0), -(\partial^0 A^3 - \partial^3 A^0)).$$

b) (*Dorian*) By inspection, the electromagnetic field tensor can be written as [8]

$$T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

c) (*Dimitrios*) When the two indices are flipped, we have

$$T_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

because the tensor is antisymmetric. When $\mu = \nu$, we have

$$T_{\nu\nu} = \partial_\nu A_\nu - \partial_\nu A_\nu = 0.$$

The fact that $T_{\nu\nu} = 0$ accounts for 4 components, the ones on the diagonal, and $T_{\nu\mu} = -T_{\mu\nu}$ additionally implies that we only need to find 6 elements of $T_{\mu\nu}$ in order to find the rest. $T_{\mu\nu}$ holds 16 components total.

d) (*Dimitrios*)

$$T_{21} = \partial_2 A_1 - \partial_1 A_2 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$$

$$T_{10} = \partial_1 A_0 - \partial_0 A_1 = \frac{\partial}{\partial x}\frac{\phi}{c} + \frac{\partial A_x}{c\partial t} = -\frac{E_x}{c}$$

$$T_{20} = \partial_2 A_0 - \partial_0 A_2 = \frac{\partial}{\partial y}\frac{\phi}{c} + \frac{\partial A_y}{c\partial t} = -\frac{E_y}{c}$$

[7] *The Feynman Lectures on Physics Volume II* retrieved on December 3, 2017

[8] *Lecture 13 Notes, Electromagnetic Theory II* Retrieved December 3, 2017

$$T_{30} = \partial_3 A_0 - \partial_0 A_3 = \frac{\partial}{\partial z}\frac{\phi}{c} + \frac{\partial A_z}{c\partial t} = -\frac{E_z}{c}$$

$$T_{31} = \partial_3 A_1 - \partial_1 A_3 = -\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x} = -B_y$$

$$T_{32} = \partial_3 A_2 - \partial_2 A_3 = -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} = B_x$$

$$T_{\mu\nu} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix}$$

**Solution: (*Dorian*) Moving Solenoid**

The magnetic field of the long solenoid positioned in frame F is

$$B_x = \mu_0 n I.$$

Note that $n$ represents the number of turns per unit length of the solenoid and $I$ represents the current travelling through the solenoid. Suppose that observer one is still with respect to the solenoid and that the solenoid is moving along the $x$ axes relative to observer two. Using previously derived formulas for length contraction, we have

$$n' = \gamma n.$$

Using previously derived formulas for time contraction, we have

$$I' = \frac{1}{\gamma} I.$$

Therefore,

$$B'_x = \mu_0 \gamma n \frac{1}{\gamma} I = \mu_0 n I = B_x.$$

Since the magnetic field does not change with motion, the electric field also does not change, and so we have[9]

$$E'_x = E_x.$$

**Solution: (*Dorian*) Correction for Maxwell's Equations**

The equation

$$\vec{\nabla} \times \vec{B} = uJ$$

becomes inaccurate when

$$\frac{\partial \vec{E}}{\partial t}$$

is not negligible. For example, a capacitor that discharges through a resistor does not have a constant electric field.[10]

[9]Griffiths, D. J. (1999). *Introduction to Electrodynamics.* Upper Saddle River, NJ: Prentice Hall.
[10]Griffiths, D. J. (1999). *Introduction to Electrodynamics.* Upper Saddle River, NJ: Prentice Hall.