Lagrangians and Hamiltonians

A Brief Introduction

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Lagrangians

When Should the Lagrangian Be Used

Since the Lagrangian is consistent with Newtonian mechanics, a Lagrangian can always be used to solve problems in classical mechanics. However, in almost any semi-complicated problem, a Lagrangian reformulation simplifies work greatly. The Lagrangian is particularly important to master because its symmetric properties yield it essential in quantum mechanics and modern physics.

Virtual Displacement and Work

Virtual Displacement: Infinitesimal change of coordinates $\delta\mathbf{r}_i$ at some time t

We restrict ourselves to conservative forces - nonconservative forces such as friction are the result of macroscopic interactions and are thus pseudo forces. Thus, we lose little in this restriction.

Principle of Virtual Work

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0$$

Generalized Force

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Lagrange–d'Alembert Principle

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Easy Mistake: "In order to minimize the action, let's take the derivative of the action with respect to t. By the fundamental theorem of calculus, this is $\mathcal{L}(t_1) - \mathcal{L}(t_0)$. To find the critical points, we set $\mathcal{L}(t_1) - \mathcal{L}(t_0) = 0$. One of the solutions minimize the path."

Why is this wrong?

Derivation of the Euler-Lagrange Equations

Consider

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt$$

Let $q(t)$ minimize the action. Then $\delta q(t_1) = \delta q(t_2) = 0$. The difference in action of a transform of $q \mapsto q + \delta q$ is

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt = 0$$

Derivation of the Euler-Lagrange Equations

Effecting the variation, we have

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

Integrating the second term by parts with $\delta \dot{q} = \frac{d\delta q}{dt}$, we have

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \dot{\delta}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q dt = 0$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

The Principle of Least Action

Define the action to be

$$S \equiv \int_{t_0}^{t_1} \underbrace{T - V}_{\text{Lagrangian}} \, dt$$

where T is kinetic energy and V is potential energy.

Key Concept: Objects move along paths which *minimize* the action.

Single Particle

Show the equivalence of Newton's equations of motion with the Lagrangian in the case of a simple particle in space, using cartesian coordinates.

Atwood's Machine

Use the Lagrangian to derive the acceleration of the conservative system of the Atwood machine with holonomic, scleronomous constraints

Mass-Spring Problem

Derive Hooke's law using the Euler-Lagrange equations

Pendulum Problem (Morin)

Consider a pendulum made of a spring with a mass m on the end. The spring is arranged to lie in a straight line (which we can arrange by, say, wrapping the spring around a rigid massless rod). The equilibrium length of the spring is l. Let the spring have length $l + x(t)$, and let its angle with the vertical be $\theta(t)$. Assuming that the motion takes place in a vertical plane, find the equations of motion for x and .

Noether’s Theorem

Every differentiable symmetry of the action of a physical system has a corresponding conservation law.

Extension to Non-conservative forces

Rayleigh's Dissipation Function:

$$\mathcal{F} = \frac{1}{2}\sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

Hamiltonians

Definition

Unlike the Lagrangian, the Hamiltonian is defined to be the sum of kinetic energy and potential energy. It describes the first-order equations of motion and can be solved in a set of 2n, coupled, first-order differential equations

Derivation via the Legendre Transformation

Recall that the Lagrangian is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

In order to find the Hamiltonian, we need to change variables from $(q, \dot{q}, t)$ to (q, p, t). We can do this via the Legendre Transformation.

Derivation via the Legendre Transformation

Find the differential of the Lagrangian, $\mathcal{L}\left(q, \dot{q}, t\right)$

Derivation via the Legendre Transformation

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$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

Note that momentum is defined as $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$. Substitute momentum into the Lagrange equation to get the differential.

Legendre Transformation

$$d\mathcal{L} = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

The Hamiltonian is generated by the Legendre transformation.

$$H(q, p, t) = \dot{q}_i p_i \mathcal{L}(q, \dot{q}, t)$$

Find the Differential of the Hamiltonian

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial \mathcal{L}}{\partial t}$$

Note that we can also write

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

Canonical Hamiltonian Equation

Conclusion

Importance

Lagrangians and Hamiltonians are critical in quantum mechanics